

A CLASS OF SASAKIAN 5-MANIFOLDS

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ABSTRACT. We obtain some general results on Sasakian Lie algebras and prove as a consequence that a $(2n + 1)$ -dimensional nilpotent Lie group admitting left-invariant Sasakian structures is isomorphic to the real Heisenberg group H_{2n+1} . Furthermore, we classify Sasakian Lie algebras of dimension 5 and determine which of them carry a Sasakian α -Einstein structure. We show that a 5-dimensional solvable Lie group with a left-invariant Sasakian structure and which admits a compact quotient by a discrete subgroup is isomorphic to either H_5 or a semidirect product $\mathbb{R} \ltimes (H_3 \times \mathbb{R})$. In particular, the compact quotient is an S^1 -bundle over a 4-dimensional Kähler solvmanifold.

1. INTRODUCTION

A Sasakian structure is the analogous in odd dimensions of a Kähler structure. Indeed, by [5] a Riemannian manifold (M, g) of odd dimension $2n+1$ admits a compatible Sasakian structure if and only if the Riemannian cone $M \times \mathbb{R}^+$ is Kähler.

In dimension 3 a homogeneous Sasakian manifold has to be a Lie group endowed with a left-invariant Sasakian structure by [27]. Therefore the classification of 3-dimensional Sasakian homogeneous spaces depends on the classification of 3-dimensional Sasakian Lie algebras.

By [26] a compact, simply connected, 5-dimensional homogeneous contact manifold is diffeomorphic to the 5-dimensional sphere S^5 or to the product of two spheres $S^2 \times S^3$. Moreover, both S^5 and $S^2 \times S^3$ carry Sasakian-Einstein structures (see [7, 24]). Other explicit examples of Sasakian-Einstein 5-manifolds have been found in [16], while toric Sasakian manifolds in dimension 5 have been studied in [9] and [11]. A classification of Sasakian-Einstein 5-manifolds of cohomogeneity 1 has been obtained in [12] and a classification of 5-dimensional Lie groups endowed with a left-invariant contact structure was obtained in [14].

As far as we know in the literature the only result about 5-dimensional Lie groups admitting left-invariant Sasakian structures is in the case of nilpotent Lie groups. Indeed, in [32] it was shown that the only 5-dimensional nilpotent Sasakian Lie algebra is the real Heisenberg Lie algebra \mathfrak{h}_5 . It was proved in [9] that the real Heisenberg Lie group H_{2n+1} admits no bi-invariant Sasakian structure, even if it has a left-invariant Sasakian structure as well as a right-invariant one. These two structures are different.

The aim of this paper is to classify 5-dimensional Lie groups endowed with a left-invariant Sasakian structure. This is equivalent to determining all 5-dimensional Sasakian

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Lie algebras. The study of Sasakian Lie algebras is done taking into account the center of the Lie algebra, which can only be trivial or 1-dimensional. In this way we obtain some general results on Sasakian Lie algebras in any dimensions. In particular, we show that the only $(2n + 1)$ -dimensional nilpotent Lie algebra admitting a Sasakian structure is the real Heisenberg Lie algebra \mathfrak{h}_{2n+1} .

By using the previous general results and independently from the list obtained in [14], we obtain a classification up to isomorphism of 5-dimensional Sasakian Lie algebras.

Main Theorem. *Let \mathfrak{g} be a 5-dimensional Lie algebra admitting a Sasakian structure. Then*

- *if \mathfrak{g} has non-trivial center $\mathfrak{z}(\mathfrak{g})$, then \mathfrak{g} is solvable with $\dim \mathfrak{z}(\mathfrak{g}) = 1$ and the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ carries an induced Kähler structure (see Theorem 4.1);*
- *if \mathfrak{g} has trivial center, then it is isomorphic to one of the following Lie algebras: the direct products $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$, or the non-unimodular solvable Lie algebra $\mathfrak{g}_0 \cong \mathbb{R}^2 \ltimes \mathfrak{h}_3$ with structure equations*

$$(1) \quad \begin{aligned} [e_1, e_3] &= e_3, & [e_1, e_4] &= \frac{1}{2}e_4, & [e_1, e_5] &= \frac{1}{2}e_5, \\ [e_2, e_4] &= e_5, & [e_2, e_5] &= -e_4, & [e_4, e_5] &= -e_3, \end{aligned}$$

where $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the Lie group of affine motions of \mathbb{R} and \mathfrak{h}_3 is the real 3-dimensional Heisenberg Lie algebra (see Theorem 4.4).

As a consequence we obtain that \mathfrak{g} is either solvable or a direct product of a 3-dimensional semisimple ideal with the radical $\mathfrak{aff}(\mathbb{R})$.

In the case of non-trivial center we determine the list of the 5-dimensional Sasakian Lie algebras by using the classification of 4-dimensional Kähler Lie algebras given by Ovando in [25].

Moreover, we prove that the only 5-dimensional simply connected Lie groups with a left-invariant Sasakian structure which admit a compact quotient by a discrete subgroup are the real Heisenberg group H_5 or a semidirect product $\mathbb{R} \ltimes (\mathbb{R} \times H_3)$ (see Corollary 4.2). By [21] a solvmanifold, i.e. a compact quotient of a solvable Lie group by a discrete subgroup, endowed with a Kähler structure is a finite quotient of a complex torus. We show that a compact quotient of a 5-dimensional solvable Lie group with a left-invariant Sasakian structure by a uniform discrete subgroup is an S^1 -bundle over a 4-dimensional Kähler solvmanifold.

By [1, Proposition 4.2] \mathfrak{g}_0 is the only solvable (non-nilpotent) 5-dimensional Lie algebra admitting a Sasakian α -Einstein structure. We show that a 5-dimensional Sasakian α -Einstein Lie algebra is isomorphic either to \mathfrak{h}_5 , \mathfrak{g}_0 or to $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$.

Moreover, by [15] it is known that a Lie algebra of dimension at least 5 cannot admit a Sasakian-Einstein structure.

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2. PRELIMINARIES

A triple (Φ, α, ξ) on a $(2n + 1)$ -dimensional manifold M is an *almost contact structure* if ξ is a nowhere vanishing vector field, α is a 1-form, and Φ is a tensor of type $(1, 1)$ such

that

$$(2) \quad \alpha(\xi) = 1, \quad \Phi^2 = -I + \xi \otimes \alpha.$$

The vector field ξ defines the characteristic foliation \mathcal{F} with 1-dimensional leaves, and the kernel of α defines the codimension one sub-bundle $\mathcal{D} = \ker \alpha$. Then there is the canonical splitting of the tangent bundle TM of M

$$TM = \mathcal{D} \oplus \mathcal{L},$$

where \mathcal{L} is the trivial line bundle generated by ξ . Note that conditions (2) imply

$$(3) \quad \Phi(\xi) = 0, \quad \alpha \circ \Phi = 0.$$

If the 1-form α satisfies the condition

$$\alpha \wedge (d\alpha)^n \neq 0,$$

then the subbundle \mathcal{D} defines a *contact structure* on M . In this case the vector field ξ is called the *Reeb vector field* and α is called a *contact form*. Contact structures can be considered as the odd-dimensional counterpart of symplectic structures. If α is a contact form, then the associated Reeb vector field satisfies

$$(4) \quad d\alpha(\xi, X) = 0$$

for any vector field X on M .

Similarly to the case of an almost complex structure, there is the notion of integrability of an almost contact structure. Indeed, an almost contact structure (Φ, α, ξ) is called *normal* if the Nijenhuis tensor N_Φ associated to the tensor Φ defined by

$$(5) \quad N_\Phi(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y],$$

satisfies the condition

$$N_\Phi = -d\alpha \otimes \xi.$$

This last condition is equivalent to requiring that the almost complex structure

$$(6) \quad J \left(X, f \frac{\partial}{\partial t} \right) = \left(\Phi X - f\xi, \alpha(X) \frac{\partial}{\partial t} \right)$$

on the product $M \times \mathbb{R}$ be integrable, where f is a smooth function on $M \times \mathbb{R}$ and t is the coordinate on \mathbb{R} (see [30]).

A Riemannian metric g on an almost contact manifold (M, Φ, α, ξ) is *compatible* with the almost contact structure if

$$g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\alpha(Y),$$

for any vector fields X, Y . In this case the structure (Φ, α, ξ, g) is called an *almost contact metric structure*. Any almost contact structure admits a compatible metric.

An almost contact metric structure (Φ, α, ξ, g) is said to be *contact metric* if

$$2g(X, \Phi Y) = d\alpha(X, Y).$$

In this case α is a contact form and we denote

$$\omega(X, Y) = g(X, \Phi Y).$$

Definition 2.1. [28] *A Sasakian structure is a normal contact metric structure, i.e. an almost contact metric structure (Φ, α, ξ, g) such that*

$$N_\Phi = -d\alpha \otimes \xi, \quad d\alpha = 2\omega.$$

A Sasakian structure can be also characterized in terms of the Riemannian cone over the manifold. More precisely, we recall that a Riemannian manifold (M, g) admits a compatible Sasakian structure if and only if the cone $M \times \mathbb{R}^+$ equipped with the metric $h = t^2g + dt \otimes dt$ is Kähler (see for instance [5]). Furthermore, in this case the Reeb vector field is Killing and the covariant derivative of Φ with respect to the Levi-Civita connection of g is given by

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi$$

for any pair of vector fields X and Y on M (see for instance [5]).

If a $(2n + 1)$ -dimensional manifold M admits a Sasakian structure, the product metric on $M \times \mathbb{R}$ is compatible with the complex structure J given by (6) and moreover by [33, Proposition 3.5] the corresponding Hermitian structure is locally conformally Kähler with parallel Lee form.

A contact metric structure (Φ, α, ξ, g) is Sasakian if and only if its Riemannian curvature tensor satisfies the condition

$$R(X, Y)\xi = \alpha(Y)X - \alpha(X)Y,$$

for any vector fields X and Y (see for instance [2]). This implies that the Ricci tensor Ric_g associated to a Sasakian metric satisfies

$$\text{Ric}_g(\xi, X) = 2n \alpha(X)$$

for any vector field X on M , where $\dim M = 2n + 1$. In particular $\text{Ric}_g(\xi, \xi) = 2n$ and the metric g is never Ricci-flat.

When the Ricci curvature tensor is given by $\text{Ric}_g = \lambda g + \nu \alpha \otimes \alpha$, for some constants $\lambda, \nu \in \mathbb{R}$, the Sasakian structure is called α -Einstein ([23, 29, 8]). In his original definition Okumura assumed that both λ and ν are functions, and then he showed, as for the Einstein metrics, that they must be constant when the dimension of the manifold is greater than three. When $\nu = 0$, the Sasakian structure is called Sasakian-Einstein. These structures have been intensively studied by many authors (see for instance [6, 7, 16, 17] and references therein). Finally, we recall that a 5-dimensional manifold is Sasakian α -Einstein if and only if it is Sasakian-hypo (see [13]).

3. SASAKIAN LIE ALGEBRAS

In this section we will begin our study of left-invariant Sasakian structures on Lie groups. Such a structure corresponds to a Sasakian structure on the associated Lie algebra.

Definition 3.1. *A Sasakian structure on a Lie algebra \mathfrak{g} is a quadruple (Φ, α, ξ, g) , where $\Phi \in \text{End}(\mathfrak{g})$, $\alpha \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and g is an inner product on \mathfrak{g} such that*

$$\begin{aligned} \alpha(\xi) &= 1, \quad \Phi^2 = -I + \xi \otimes \alpha, \quad g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\alpha(Y), \\ 2g(X, \Phi Y) &= d\alpha(X, Y), \quad N_\Phi = -d\alpha \otimes \xi, \end{aligned}$$

where N_Φ is defined as in (5). A Lie algebra equipped with a Sasakian structure will be called a Sasakian Lie algebra. The vector ξ will be called the Reeb vector.

Remark 3.2. We note that, in this setting, formula (4) reads

$$\alpha(\xi, X) = 0$$

for any $X \in \mathfrak{g}$.

Example 3.3. The classical example of a Sasakian Lie algebra is given by the $(2n + 1)$ -dimensional real Heisenberg Lie algebra \mathfrak{h}_{2n+1} . We recall that $\mathfrak{h}_{2n+1} = \text{Span}\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$, where

$$[X_i, Y_i] = Z,$$

for $i = 1, \dots, n$; in this case, a Sasakian structure is defined by

$$\Phi(X_i) = Y_i, \quad \Phi(Y_i) = -X_i, \quad \Phi(Z) = 0, \quad i = 1, \dots, n,$$

the inner product g is obtained by declaring the basis above orthonormal, $\xi = Z$ and α is the dual 1-form of Z with respect to the metric g .

In general for a Lie algebra \mathfrak{g} with a contact structure α we can prove the following property for its center $\mathfrak{z}(\mathfrak{g})$.

Proposition 3.4. *Let (\mathfrak{g}, α) be a contact Lie algebra with ξ its Reeb vector and let $\mathfrak{z}(\mathfrak{g})$ be the center of \mathfrak{g} . Then*

1. $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$;
2. if $\dim \mathfrak{z}(\mathfrak{g}) = 1$, then $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\xi$.

Proof. The first item is well known and follows from the fact that $d\alpha$ is non-degenerate on $\ker \alpha$. For the second item we fix an arbitrary generator Z of $\mathfrak{z}(\mathfrak{g})$. We may write $Z = a\xi + X$, where $a \in \mathbb{R}$ and $X \in \ker \alpha$. We have

$$0 = \alpha([Z, Y]) = \alpha([a\xi + X, Y]) = \alpha([X, Y]) = -d\alpha(X, Y)$$

for all $Y \in \ker \alpha$. By the non-degeneracy of $d\alpha$ on $\ker \alpha$, it follows that $X = 0$ and, consequently, $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\xi$. \square

Remark 3.5. We recall that due to [4, Theorem 5], the only semisimple Lie algebras admitting a contact form are $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$.

3.1. Non-trivial center. We show that in the case of Sasakian Lie algebras with non-trivial center the kernel of the contact form inherits a natural structure of Kähler Lie algebra. Moreover two Sasakian Lie algebras are isomorphic if and only if the corresponding Kähler Lie algebras are equivalent. This allows us to use the classification of the 4-dimensional Kähler Lie algebras of [25] to classify 5-dimensional Sasakian Lie algebras with non-trivial center.

We start by considering the following

Proposition 3.6. *Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra with non-trivial center $\mathfrak{z}(\mathfrak{g})$ generated by ξ . Then the quadruple $(\ker \alpha, \theta, \Phi, g)$ is a Kähler Lie algebra, where θ is the component of the Lie bracket of \mathfrak{g} on $\ker \alpha$.*

Proof. Let X, Y, Z in $\ker \alpha$, then

$$\begin{aligned}
0 &= [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] \\
&= [X, \theta(Y, Z) + \alpha([Y, Z])\xi] + [Z, \theta(X, Y) + \alpha([X, Y])\xi] \\
&\quad + [Y, \theta(Z, X) + \alpha([Z, X])\xi] \\
&= [X, \theta(Y, Z)] + [Z, \theta(X, Y)] + [Y, \theta(Z, X)] \\
&= \theta(X, \theta(Y, Z)) + \theta(Z, \theta(X, Y)) + \theta(Y, \theta(Z, X)) \\
&\quad + \alpha([X, \theta(Y, Z)] + [Z, \theta(X, Y)] + [Y, \theta(Z, X)])\xi
\end{aligned}$$

i.e.

$$\theta(X, \theta(Y, Z)) + \theta(Z, \theta(X, Y)) + \theta(Y, \theta(Z, X)) = 0, \quad d\omega = 0,$$

2ω being the restriction of $d\alpha$ on $\ker \alpha$. Then θ defines a Lie bracket on $\ker \alpha$. Since ω is non-degenerate, the statement holds. \square

Conversely, let $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \omega, \Phi, g)$ be a Kähler Lie algebra and set $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}\xi$. Then defining

$$[X, Y] = [X, Y]_{\mathfrak{h}} - \omega(X, Y)\xi$$

for $X, Y \in \mathfrak{h}$ and

$$[\xi, \mathfrak{h}] = 0$$

we obtain a new Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ equipped with a natural Sasakian structure, where the contact form α on \mathfrak{g} is defined as

$$\alpha(a\xi + X) = a$$

for all $X \in \mathfrak{h}$ and Φ and g are extended in a natural way.

Corollary 3.7. *Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra with non-trivial center $\mathfrak{z}(\mathfrak{g})$ generated by ξ . Then $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ inherits a natural Kähler structure.*

Now we have the following easy-to-prove proposition which will be used in §4.

Proposition 3.8. *Two Sasakian Lie algebras with non-trivial center $(\mathfrak{g}_1, \Phi_1, \alpha_1, \xi_1, g_1)$, $(\mathfrak{g}_2, \Phi_2, \alpha_2, \xi_2, g_2)$ are isomorphic if and only if $\ker \alpha_1$ and $\ker \alpha_2$ are isomorphic as Kähler Lie algebras.*

Since a nilpotent Lie algebra has always non-trivial center, we can apply the results above in order to determine all nilpotent Lie algebras admitting a Sasakian structure. It is known that in dimensions 3 and 5 the only nilpotent Sasakian Lie algebras are the real Heisenberg algebras \mathfrak{h}_3 and \mathfrak{h}_5 , respectively (see [18] and [32, Corollary 5.5]). We show next that this still holds in any dimension.

Theorem 3.9. *Let \mathfrak{g} be a $(2n+1)$ -dimensional nilpotent Lie algebra admitting a Sasakian structure. Then \mathfrak{g} is isomorphic to the $(2n+1)$ -dimensional Heisenberg Lie algebra.*

Proof. Let (Φ, α, ξ, g) be a Sasakian structure on \mathfrak{g} . Since \mathfrak{g} is nilpotent it has non-trivial center $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\xi$. The quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is a Kähler nilpotent Lie algebra, hence it is unimodular, and therefore, using a result of Hano [22], it is flat. As a consequence, $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is abelian. This implies immediately that \mathfrak{g} is isomorphic to the Heisenberg Lie algebra. \square

Remark 3.10. According to [31] the $(2n+1)$ -dimensional Heisenberg Lie algebra admits a contact Calabi-Yau structure. In particular \mathfrak{h}_{2n+1} admits Sasakian α -Einstein structures.

3.2. Trivial center. In the case the Sasakian Lie algebra \mathfrak{g} has trivial center, we have the following properties for ad_ξ .

Proposition 3.11. *Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra. Then*

1. $\text{ad}_\xi \Phi = \Phi \text{ad}_\xi$, and therefore $\ker \text{ad}_\xi$ and Im ad_ξ are Φ -invariant subspaces of \mathfrak{g} ;
2. $\text{ad}_\xi \Phi$ is symmetric with respect to g ;
3. ad_ξ is skew-symmetric with respect to g and therefore $(\text{Im ad}_\xi)^\perp = \ker \text{ad}_\xi$.

Proof. The first item is an easy consequence of $N_\Phi = -d\alpha \otimes \xi$ and (3). Indeed, for any $X \in \mathfrak{g}$,

$$\begin{aligned} 0 = N_\Phi(\xi, X) &= [\Phi\xi, \Phi X] - \Phi[\Phi\xi, X] - \Phi[\xi, \Phi X] + \Phi^2[\xi, X] \\ &= -\Phi[\xi, \Phi X] - [\xi, X], \end{aligned}$$

i.e.

$$[\xi, \Phi X] = \Phi[\xi, X].$$

The second item follows from the following computation, for $X, Y \in \mathfrak{g}$,

$$\begin{aligned} 2g([\xi, \Phi X], Y) &= 2g(\Phi[\xi, X], Y) = -2g([\xi, X], \Phi Y) = -d\alpha([\xi, X], Y) \\ &= \alpha([\xi, X], Y) = -\alpha([X, Y], \xi) - [\xi, Y], X) \\ &= \alpha([\xi, Y], X) = -d\alpha([\xi, Y], X) = -2g([\xi, Y], \Phi X) \\ &= 2g(X, [\xi, \Phi Y]). \end{aligned}$$

Finally, let $X, Y \in \mathfrak{g}$. We can write $X = a\xi + \Phi X'$, with $X' \in \ker \alpha$, and thus the third item follows from

$$\begin{aligned} g([\xi, X], Y) &= g([\xi, a\xi + \Phi X'], Y) = g([\xi, \Phi X'], Y) = g(X', [\xi, \Phi Y]) \\ &= -g(\Phi X', [\xi, Y]) = -g(X - a\xi, [\xi, Y]) = -g(X, [\xi, Y]). \end{aligned}$$

This shows also that $\ker \text{ad}_\xi \subseteq (\text{Im ad}_\xi)^\perp$. For dimensional reasons we have that $\ker \text{ad}_\xi = (\text{Im ad}_\xi)^\perp$. \square

As a direct consequence of Proposition 3.11 we have the following

Corollary 3.12. *Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra. Then there is an orthogonal decomposition*

$$\mathfrak{g} = \ker \text{ad}_\xi \oplus \text{Im ad}_\xi.$$

Proposition 3.13. *Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a Sasakian Lie algebra with trivial center.*

- (1) *If $\dim \mathfrak{g} \geq 5$, then $\ker \text{ad}_\xi$ is a Sasakian Lie subalgebra of \mathfrak{g} with non-trivial center.*
- (2) *If $X \in \ker \text{ad}_\xi, Y \in \text{Im ad}_\xi$, then $[X, Y] \in \text{Im ad}_\xi$.*

Proof. (1) Since ad_ξ is a derivation, its kernel is a Lie subalgebra of \mathfrak{g} . Furthermore, let $X \in \ker \alpha \cap \ker \text{ad}_\xi$ and $Y \in \mathfrak{g}$, then

$$d\alpha(X, \text{ad}_\xi Y) = -\alpha([X, [\xi, Y]]) = \alpha([Y, [X, \xi]]) + \alpha([\xi, [Y, X]]) = 0.$$

Hence if $d\alpha(X, Z) = 0$ for any $Z \in \ker \operatorname{ad}_\xi$, then $d\alpha(X, W) = 0$ for any $W \in \ker \alpha$ and consequently $X = 0$. It follows that the restriction of α to $\ker \operatorname{ad}_\xi$ is a contact form and ξ is still the Reeb vector. Clearly ξ belongs to the center of this subalgebra and the restrictions of Φ and g induce a Sasakian structure on $\ker \operatorname{ad}_\xi$.

(2) We can write

$$X = a\xi + X', \quad Y = [\xi, Y'],$$

with $a \in \mathbb{R}$, $X' \in \ker \operatorname{ad}_\xi \cap \ker \alpha$, $Y' \in \mathfrak{g}$. Then, by using Jacobi identity,

$$[X, Y] = a[\xi, [\xi, Y']] + [X', [\xi, Y']] = a[\xi, [\xi, Y']] - [\xi, [Y', X']],$$

which belongs to $\operatorname{Im} \operatorname{ad}_\xi$. \square

Remark 3.14. Note that if $\ker \alpha = \operatorname{Im} \operatorname{ad}_\xi$, then the commutator ideal $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ coincides with \mathfrak{g} .

Assume now that the Sasakian Lie algebra $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ with trivial center satisfies in addition the condition $\mathfrak{g}' \neq \mathfrak{g}$.

With respect to the decomposition $\mathfrak{g} = \ker \operatorname{ad}_\xi \oplus \operatorname{Im} \operatorname{ad}_\xi$, we can write

$$(\operatorname{ad}_\xi)|_{\ker \alpha} = \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}, \quad \Phi|_{\ker \alpha} = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where $U : \operatorname{Im} \operatorname{ad}_\xi \rightarrow \operatorname{Im} \operatorname{ad}_\xi$ is a non-singular operator. Therefore from the equality $\Phi \operatorname{ad}_\xi = \operatorname{ad}_\xi \Phi$ one gets that

$$B = C = 0, \quad DU = UD,$$

and since $(\Phi|_{\ker \alpha})^2 = -I$,

$$A^2 = D^2 = -I.$$

In particular, if \mathfrak{g} is solvable, then the Reeb vector ξ cannot belong to the commutator \mathfrak{g}' .

3.3. 3-dimensional Sasakian Lie algebras. Simply connected homogeneous 3-dimensional contact metric manifolds were classified by Perrone in [27], showing that the homogeneous space has to be a Lie group with a left-invariant contact metric structure. Among these Lie groups we can find the ones that admit a Sasakian structure.

For the sake of completeness we perform the classification of 3-dimensional Sasakian Lie groups even if it is already known also by [18, 10]. Indeed, Geiges in [18] has determined the diffeomorphism types of compact manifolds of dimension 3 that admit a Sasakian structure.

Theorem 3.15. *Any 3-dimensional Sasakian Lie algebra is isomorphic to one of the following: $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$, \mathfrak{h}_3 .*

Proof. It is well known that, if the commutator \mathfrak{g}' of a 3-dimensional Lie algebra \mathfrak{g} coincides with \mathfrak{g} , then \mathfrak{g} is semisimple and it is isomorphic to either $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$. Both Lie algebras admit a Sasakian structure (see for instance [18, 10, 27]).

If $\mathfrak{g}' \neq \mathfrak{g}$ then \mathfrak{g} is solvable and, by Corollary 3.12, we have the orthogonal decomposition

$$\mathfrak{g} = \ker \operatorname{ad}_\xi \oplus \operatorname{Im} \operatorname{ad}_\xi,$$

where ξ is the Reeb vector. Since $\ker \operatorname{ad}_\xi \cap \ker \alpha$ is Φ -invariant, the only possibility is that $\dim(\ker \operatorname{ad}_\xi \cap \ker \alpha) = 2$ and thus ξ belongs to the center $\mathfrak{z}(\mathfrak{g})$. The Kähler quotient

$\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is then isomorphic to \mathbb{R}^2 or $\mathfrak{aff}(\mathbb{R})$. It is easy to show that in the former case \mathfrak{g} is isomorphic to \mathfrak{h}_3 . In the latter case there exists a basis $\{e^1, e^2, e^3\}$ of \mathfrak{g}^* such that

$$de^1 = 0, \quad de^2 = e^{12}, \quad de^3 = 2e^{12},$$

with respect to which the Sasakian structure is

$$\xi = e_3, \quad \alpha = e^3, \quad \Phi(e_1) = e_2, \quad \omega = e^{12}.$$

Considering the new basis

$$E^j = e^j, \quad j = 1, 2, \quad E^3 = e^3 - 2e^2,$$

we have the isomorphism of \mathfrak{g} with $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$. \square

4. 5-DIMENSIONAL SASAKIAN LIE ALGEBRAS

In order to classify 5-dimensional Sasakian Lie algebras \mathfrak{g} up to isomorphism we will consider separately the case of Lie algebras with trivial or non-trivial center.

4.1. 5-dimensional Sasakian Lie algebras with non-trivial center. Proposition 3.8 together with the classification in [25] of 4-dimensional Kähler Lie algebras allow us to classify 5-dimensional Sasakian Lie algebras with non-trivial center.

Theorem 4.1. *Any 5-dimensional Sasakian Lie algebra \mathfrak{g} with non-trivial center is isomorphic to one of the following Lie algebras*

$$\begin{aligned} \mathfrak{g}_1 &= (0, 0, 0, 0, e^{12} + e^{34}) \simeq \mathfrak{h}_5; \\ \mathfrak{g}_2 &= (0, -e^{12}, 0, 0, e^{12} + e^{34}) \simeq \mathfrak{aff}(\mathbb{R}) \times \mathfrak{h}_3; \\ \mathfrak{g}_3 &= (0, -e^{13}, e^{12}, 0, e^{14} + e^{23}) \simeq \mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R}); \\ \mathfrak{g}_4 &= (0, -e^{12}, 0, -e^{34}, e^{12} + e^{34}) \simeq \mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}; \\ \mathfrak{g}_5 &= \left(\frac{1}{2}e^{14}, \frac{1}{2}e^{24}, -e^{12} + e^{34}, 0, e^{12} - e^{34} \right) \simeq \mathbb{R} \times (\mathbb{R} \ltimes \mathfrak{h}_3); \\ \mathfrak{g}_6 &= (2e^{14}, -e^{24}, -e^{12} + e^{34}, 0, e^{23}) \simeq \mathbb{R} \ltimes \mathfrak{n}_4; \\ \mathfrak{g}_7^\delta &= \left(\frac{\delta}{2}e^{14} + e^{24}, -e^{14} + \frac{\delta}{2}e^{24}, -e^{12} + \delta e^{34}, 0, e^{12} - \delta e^{34} \right) \simeq \mathbb{R} \times (\mathbb{R} \ltimes \mathfrak{h}_3), \quad \delta > 0; \\ \mathfrak{g}_8^\delta &= (e^{14}, \delta e^{34}, -\delta e^{24}, 0, e^{14} + e^{23}) \simeq \mathbb{R} \ltimes_\delta (\mathfrak{h}_3 \times \mathbb{R}), \quad \delta > 0. \end{aligned}$$

Proof. By Corollary 3.7, if $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ is a 5-dimensional Sasakian Lie algebra, then the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is a 4-dimensional Kähler Lie algebra with the Kähler structure induced by the Sasakian one. We may choose a basis $\{e_1, \dots, e_5\}$ of \mathfrak{g} such that

$$\xi = e_5, \quad \alpha = e^5, \quad \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \text{Span}\{e_1, e_2, e_3, e_4\}$$

and $de^5 = 2\Omega$, where Ω is the Kähler form on the quotient.

4-dimensional Kähler Lie algebras and their possible Kähler forms have been classified in [25]. By using this classification we obtain that \mathfrak{g} is isomorphic to one of the following Lie algebras

$$\begin{aligned}
\mathfrak{k}_1 &= (0, 0, 0, 0, \lambda e^{12} + \mu e^{34}), \quad \lambda, \mu < 0; \\
\mathfrak{k}_2 &= (0, -e^{12}, 0, 0, \lambda e^{12} + \mu e^{34}), \quad \lambda, \mu < 0; \\
\mathfrak{k}_3 &= (0, -e^{13}, e^{12}, 0, \lambda e^{14} + \mu e^{23}), \quad \lambda, \mu < 0; \\
\mathfrak{k}_4 &= (0, -e^{12}, 0, -e^{34}, \lambda e^{12} + \mu e^{34}), \quad \lambda, \mu < 0; \\
\mathfrak{k}_5 &= \left(\frac{1}{2} e^{14}, \frac{1}{2} e^{24}, -e^{12} + e^{34}, 0, \lambda(e^{12} - e^{34}) \right), \quad \lambda < 0; \\
\mathfrak{k}_6 &= (2e^{14}, -e^{24}, -e^{12} + e^{34}, 0, \lambda e^{14} + \mu e^{23}), \quad \lambda, \mu < 0; \\
\mathfrak{k}_7^- &= \left(\frac{\delta}{2} e^{14} + e^{24}, -e^{14} + \frac{\delta}{2} e^{24}, -e^{12} + \delta e^{34}, 0, \lambda(e^{12} - \delta e^{34}) \right), \quad \delta > 0, \lambda < 0; \\
\mathfrak{k}_7^+ &= \left(\frac{\delta}{2} e^{14} + e^{24}, -e^{14} + \frac{\delta}{2} e^{24}, -e^{12} + \delta e^{34}, 0, \lambda(e^{12} - \delta e^{34}) \right), \quad \delta > 0; \\
\mathfrak{k}_8^- &= (e^{14}, \delta e^{34}, -\delta e^{24}, 0, \lambda e^{14} + \mu e^{23}), \quad \delta, \lambda > 0, \mu < 0; \\
\mathfrak{k}_8^+ &= (e^{14}, \delta e^{34}, -\delta e^{24}, 0, \lambda e^{14} + \mu e^{23}), \quad \delta, \lambda, \mu > 0;
\end{aligned}$$

equipped with the Sasakian structure $(\Phi, \alpha = e^5, \xi = e_5, g)$, where Φ is given respectively by

$$\begin{aligned}
\Phi_1(e_1) &= e_2, & \Phi_1(e_3) &= e_4 \\
\Phi_2(e_1) &= e_2, & \Phi_2(e_3) &= e_4 \\
\Phi_3(e_1) &= e_4, & \Phi_3(e_2) &= e_3 \\
\Phi_4(e_1) &= e_2, & \Phi_4(e_3) &= e_4 \\
\Phi_5(e_1) &= e_2, & \Phi_5(e_4) &= e_3 \\
\Phi_6(e_2) &= e_3, & \Phi_6(e_4) &= -2e_1 \\
\Phi_7^-(e_1) &= e_2, & \Phi_7^-(e_4) &= e_3 \\
\Phi_7^+(e_1) &= -e_2, & \Phi_7^+(e_4) &= -e_3 \\
\Phi_8^-(e_4) &= e_1, & \Phi_8^-(e_2) &= e_3 \\
\Phi_8^+(e_4) &= e_1, & \Phi_8^+(e_3) &= e_2
\end{aligned}$$

and $2g(\cdot, \cdot) = de^5(\Phi \cdot, \cdot)$.

By a direct computation one may check the following isomorphisms

$$\mathfrak{k}_i \cong \mathfrak{g}_i, \quad i = 1, \dots, 5, \quad \mathfrak{k}_6^\pm \cong \mathfrak{g}_6^\tau \text{ (with } \tau = \lambda/\mu), \quad \mathfrak{k}_j^\pm \cong \mathfrak{g}_j^\delta, \quad j = 7, 8.$$

For the family

$$\mathfrak{g}_6^\tau = (2e^{14}, -e^{24}, -e^{12} + e^{34}, 0, \tau e^{14} + e^{23}), \quad \tau > 0$$

we can show that every element in the family is isomorphic to \mathfrak{g}_6 by considering the new basis

$$E_1 = 2e_1 + \tau e_5, \quad E_j = e_j, \quad j = 2, \dots, 5.$$

By Proposition 3.8, the Lie algebras \mathfrak{g}_i and \mathfrak{g}_j^δ are not isomorphic for any i and j . Moreover, \mathfrak{g}_i (respectively \mathfrak{g}_i^δ) is not isomorphic to \mathfrak{g}_k (respectively \mathfrak{g}_k^δ) for any $i \neq k$.

Applying again Proposition 3.8, it follows that the Lie algebras in the family \mathfrak{g}_7^δ are not isomorphic one each other for different values of δ . The same holds for the family \mathfrak{g}_8^δ . \square

As a consequence we have the following

Corollary 4.2. *A unimodular Sasakian Lie algebra with non-trivial center is isomorphic either to the nilpotent Heisenberg Lie algebra \mathfrak{h}_5 or the solvable Lie algebra \mathfrak{g}_3 . The simply connected Lie group G_3 with Lie algebra \mathfrak{g}_3 admits a co-compact discrete subgroup Γ .*

Proof. By a direct computation one may check that the only unimodular Sasakian Lie algebras with non-trivial center are in fact $\mathfrak{g}_1 \cong \mathfrak{h}_5$ and $\mathfrak{g}_3 \cong \mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R})$. The solvable simply connected Lie group G_3 is isomorphic to \mathbb{R}^5 with the following product

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \cos(x_1)y_2 + \sin(x_1)y_3 + x_2 \\ -\sin(x_1)y_2 + \cos(x_1)y_3 + x_3 \\ Q \\ x_5 + y_5 \end{pmatrix},$$

where

$$Q = y_4 + x_1y_5 - \sin^2(x_1)y_2y_3 - \frac{1}{4}\sin(2x_1)(y_2^2 - y_3^2) + x_2(-\sin(x_1)y_2 + \cos(x_1)y_3) + x_4.$$

The discrete subgroup

$$\Gamma = \left\{ \left(2\pi m_1, m_2, m_3, m_4, \frac{1}{2\pi}m_5 \right) \mid m_i \in \mathbb{Z} \right\}$$

acts freely and properly discontinuously on G_3 . Moreover, the quotient manifold $\Gamma \backslash G_3$ is compact. \square

The solvmanifold $\Gamma \backslash G_3$ is by construction the total space of an S^1 -bundle over a 4-dimensional non completely solvable Kähler solvmanifold. This class of Kähler solvmanifolds was found by Hasegawa in [19] (see also [20]); all of these solvmanifolds are hyper-elliptic surfaces. We recall that a solvmanifold is called completely solvable if the adjoint representation of the corresponding solvable Lie group has only real eigenvalues.

4.2. 5-dimensional Sasakian Lie algebras with trivial center. Let $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ be a 5-dimensional Sasakian Lie algebra with trivial center and $\mathfrak{g}' = \mathfrak{g}$. By [14, Section 5] the only contact Lie algebra with the above property is the semidirect product $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, where $\mathfrak{sl}(2, \mathbb{R})$ acts on \mathbb{R}^2 by matrix multiplication. We can prove the following

Proposition 4.3. *The Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ does not admit any Sasakian structure.*

Proof. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathfrak{sl}(2, \mathbb{R})$ with Lie brackets

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1$$

and let $\{e_4, e_5\}$ be the canonical basis of \mathbb{R}^2 . Then $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ has structure equations

$$\begin{aligned} de^1 &= -e^{23}, \\ de^2 &= -2e^{12}, \\ de^3 &= 2e^{13}, \\ de^4 &= -e^{14} - e^{25}, \\ de^5 &= e^{15} - e^{34}. \end{aligned}$$

A 1-form $\alpha = \sum_{i=1}^5 a_i e^i$ is contact if and only if the real numbers a_i satisfy the condition

$$(7) \quad \Delta := a_3 a_4^2 - a_2 a_5^2 - a_1 a_4 a_5 \neq 0.$$

In this case, the corresponding Reeb vector is given by

$$\xi = -\frac{1}{3\Delta} (a_4 a_5 e_1 + a_5^2 e_2 - a_4^2 e_3 + (a_1 a_5 - 2a_3 a_4) e_4 + (a_1 a_4 + 2a_2 a_5) e_5).$$

One can check that $X \in \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ belongs to $\ker \text{ad}_\xi \cap \ker \alpha$ if and only

$$X = t(-a_5 e_4 + a_4 e_5), \quad t \in \mathbb{R}.$$

If there exists a Sasakian structure with contact form α , then $\ker \text{ad}_\xi \cap \ker \alpha$ has to be Φ -invariant (see Proposition 3.11) and this is only possible if $a_4 = a_5 = 0$ in contrast with (7). □

Now we can consider the case of 5-dimensional Sasakian Lie algebras with trivial center and such that $\mathfrak{g}' \neq \mathfrak{g}$. In this case

$$\dim \ker(\text{ad}_\xi)|_{\ker \alpha} = \dim \text{Im}(\text{ad}_\xi) = 2.$$

It is easy to see that there exists an orthonormal basis $\{e_1, \dots, e_4\}$ of $\ker \alpha$ with respect to which $\Phi|_{\ker \alpha}$ can be written as

$$\Phi|_{\ker \alpha} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $\ker \text{ad}_\xi = \text{Span}\{\xi, e_1, e_2\}$, $\text{Im} \text{ad}_\xi = \text{Span}\{e_3, e_4\}$. Moreover in this basis

$$(\text{ad}_\xi)|_{\ker \alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}.$$

Note that in terms of $\{e_1, \dots, e_4\}$ the 2-form $d\alpha$ takes the standard form $d\alpha = 2(e^{12} + e^{34})$. Furthermore, taking into account that $\alpha([e_3, e_4]) = -d\alpha(e_3, e_4) = -2$, and recalling that

$\theta : \mathfrak{g} \times \mathfrak{g} \rightarrow \ker \alpha$ denotes the projection of the bracket on \mathfrak{g} onto $\ker \alpha$, we have

$$\begin{aligned}
0 &= [\xi, [e_3, e_4]] + [e_4, [\xi, e_3]] - [e_3, [\xi, e_4]] \\
&= [\xi, \theta(e_3, e_4) - 2\xi] + [e_4, a e_3 + b e_4] - [e_3, -b e_3 + a e_4] \\
&= [\xi, \theta(e_3, e_4)] + a [e_4, e_3] - a [e_3, e_4] \\
&= [\xi, \theta(e_3, e_4)] - 2a [e_3, e_4] \\
&= [\xi, \theta(e_3, e_4)] - 2a \theta(e_3, e_4) + 4a \xi
\end{aligned}$$

which implies

$$(8) \quad a = 0, \quad \theta(e_3, e_4) \in \ker \operatorname{ad}_\xi \cap \ker \alpha.$$

From now on we set $e_5 = \xi$ and denote by $\{e^1, \dots, e^5\}$ the dual basis of $\{e_1, \dots, e_5\}$. Since $b \neq 0$, by a suitable rescaling of α and ξ we may assume $b = \pm 1$. We will examine separately the two cases: Case A: $b = 1$ and Case B: $b = -1$.

We start by considering the case A. Taking into account (8) and Proposition 3.13, we can write

$$\begin{aligned}
de^1 &= a_1 e^{12} + a_6 e^{34}, \\
de^2 &= b_1 e^{12} + b_6 e^{34}, \\
(9) \quad de^3 &= -e^{45} + c_2 e^{13} + c_3 e^{14} + c_4 e^{23} + c_5 e^{24}, \\
de^4 &= e^{35} + f_2 e^{13} + f_3 e^{14} + f_4 e^{23} + f_5 e^{24}, \\
de^5 &= 2(e^{12} + e^{34}).
\end{aligned}$$

From the condition $N_\Phi = -de^5 \otimes e_5$ we get

$$c_5 = c_2 - f_3 - f_4, \quad f_5 = f_2 + c_3 + c_4.$$

Now we impose $d^2 = 0$. From the vanishing of the coefficients of e^{ij5} , $i, j = 1, \dots, 4$, in $d^2 e^k = 0$, $k = 1, \dots, 5$, we get the following linear equations

$$(10) \quad c_2 - f_3 = f_2 + c_3 = 0.$$

Moreover,

$$d^2 e^5 = (a_6 + 2c_4 + f_2 + c_3)e^{234} + (-b_6 + c_2 + f_3)e^{134}.$$

and therefore in addition to (10) we have

$$a_6 = -2c_4, \quad b_6 = 2c_2.$$

In this way the structure equations (9) of \mathfrak{g} reduce to

$$\begin{aligned}
de^1 &= a_1 e^{12} - 2c_4 e^{34}, \\
de^2 &= b_1 e^{12} + 2c_2 e^{34}, \\
de^3 &= -e^{45} + c_2 e^{13} + c_3 e^{14} + c_4 e^{23} - f_4 e^{24}, \\
de^4 &= e^{35} - c_3 e^{13} + c_2 e^{14} + f_4 e^{23} + c_4 e^{24}, \\
de^5 &= 2(e^{12} + e^{34})
\end{aligned}$$

with the structure constants satisfying the conditions

$$\begin{aligned}
c_2(a_1 + 2c_4) &= 0, & c_4(a_1 + 2c_4) &= 0, & c_2(-b_1 + 2c_2) &= 0, \\
c_4(-b_1 + 2c_2) &= 0, & a_1 c_3 - b_1 f_4 + 2 &= 0, & c_2 a_1 + c_4 b_1 &= 0.
\end{aligned}$$

We get the following solutions for the above system

$$\begin{aligned} \text{A1)} \quad & b_1 = c_2 = c_4 = 0, c_3 \neq 0, a_1 = -\frac{2}{c_3}, \\ \text{A2)} \quad & c_2 = c_4 = 0, b_1 \neq 0, f_4 = \frac{2 + a_1 c_3}{b_1}, \\ \text{A3)} \quad & b_1 = c_2 = 0, a_1 \neq 0, c_3 = -\frac{2}{a_1}, c_4 = -\frac{1}{2}a_1, \\ \text{A4)} \quad & b_1 \neq 0, c_2 = \frac{1}{2}b_1, c_4 = -\frac{1}{2}a_1, f_4 = \frac{2 + a_1 c_3}{b_1}. \end{aligned}$$

In the first two cases, we have that

$$\text{A1)}, \text{A2)} \simeq \mathfrak{aff}(\mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$$

where respectively

$$\text{A1)} \quad \begin{cases} \mathfrak{aff}(\mathbb{R}) \simeq \text{Span}\{f_4 e_1 - c_3 e_2, e_1 - c_3 e_5\}, \\ \mathfrak{sl}(2, \mathbb{R}) \simeq \text{Span}\{e_3, e_4, e_5\}. \end{cases}$$

and

$$\text{A2)} \quad \begin{cases} \mathfrak{aff}(\mathbb{R}) \simeq \text{Span}\{a_1 e_1 + b_1 e_2 + 2e_5, e_1 - c_3 e_5\}, \\ \mathfrak{sl}(2, \mathbb{R}) \simeq \text{Span}\{e_3, e_4, e_5\}. \end{cases}$$

In the other cases we see that

$$\text{A3)}, \text{A4)} \cong \mathbb{R}^2 \ltimes \mathfrak{h}_3$$

by using for A3) the new basis

$$\left\{ E_1 = a_1 e_1 + 2e_5, E_2 = \frac{1}{a_1} e_2, E_j = e_j, j = 3, 4, 5 \right\},$$

with $\mathbb{R}^2 = \text{Span}\{E_2, E_5\}$, $\mathfrak{h}_3 = \text{Span}\{E_1, E_3, E_4\}$ and

$$\text{ad}_{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{f_4}{a_1} \\ 0 & -\frac{f_4}{a_1} & \frac{1}{2} \end{pmatrix}, \quad \text{ad}_{E_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For A4) we may choose the new basis

$$\left\{ F_1 = a_1 e_1 + b_1 e_2 + 2e_5, F_2 = \frac{1}{b_1} e_1, F_j = e_j, j = 3, 4, 5 \right\},$$

with $\mathbb{R}^2 = \text{Span}\{F_2, F_5\}$, $\mathfrak{h}_3 = \text{Span}\{F_1, F_3, F_4\}$ and

$$\text{ad}_{F_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{c_3}{b_1} \\ 0 & \frac{c_3}{b_1} & -\frac{1}{2} \end{pmatrix}, \quad \text{ad}_{F_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now we study the case B. Taking into account (8) and Proposition 3.13, we can write

$$\begin{aligned}
 (11) \quad & de^1 = a_1 e^{12} + a_6 e^{34}, \\
 & de^2 = b_1 e^{12} + b_6 e^{34}, \\
 & de^3 = e^{45} + c_2 e^{13} + c_3 e^{14} + c_4 e^{23} + c_5 e^{24}, \\
 & de^4 = -e^{35} + f_2 e^{13} + f_3 e^{14} + f_4 e^{23} + f_5 e^{24}, \\
 & de^5 = 2(e^{12} + e^{34}).
 \end{aligned}$$

Condition $N_\Phi = -de^5 \otimes e_5$ implies the following linear equations

$$c_5 = c_2 - f_3 - f_4, \quad f_5 = f_2 + c_3 + c_4,$$

while $d^2 = 0$ gives

$$c_2 = f_3, \quad f_2 = -c_3, \quad a_6 = -2c_4, \quad f_3 = \frac{1}{2}b_6.$$

Hence the structure equations (11) of \mathfrak{g} reduces to

$$\begin{aligned}
 & de^1 = a_1 e^{12} - 2c_4 e^{34}, \\
 & de^2 = b_1 e^{12} + b_6 e^{34}, \\
 & de^3 = e^{45} + \frac{1}{2}b_6 e^{13} + c_3 e^{14} + c_4 e^{23} - f_4 e^{24}, \\
 & de^4 = -e^{35} - c_3 e^{13} + \frac{1}{2}b_6 e^{14} + f_4 e^{23} + c_4 e^{24}, \\
 & de^5 = 2(e^{12} + e^{34}),
 \end{aligned}$$

where the structure constants are related by the equations

$$\begin{aligned}
 & c_4(2c_4 + a_1) = 0, \quad b_6(a_1 + 2c_4) = 0, \quad c_4(b_1 - b_6) = 0, \quad b_6(b_6 - b_1) = 0 \\
 & b_1 c_4 + \frac{1}{2}a_1 b_6 = 0, \quad c_3 a_1 - b_1 f_4 - 2 = 0.
 \end{aligned}$$

This imposes to consider the following four cases:

$$\begin{aligned}
 \text{B1)} \quad & b_1 = 0, \quad b_6 = 0, \quad c_4 = 0, \quad c_3 = \frac{2}{a_1}, \\
 \text{B2)} \quad & b_6 = 0, \quad c_4 = 0, \quad f_4 = \frac{c_3 a_1 - 2}{b_1}, \\
 \text{B3)} \quad & b_1 = 0, \quad b_6 = 0, \quad c_3 = \frac{2}{a_1}, \quad c_4 = -\frac{1}{2}a_1, \\
 \text{B4)} \quad & b_6 = b_1, \quad c_4 = -\frac{1}{2}a_1, \quad f_4 = \frac{c_3 a_1 - 2}{b_1}.
 \end{aligned}$$

In the first two cases we have

$$\text{B1)}, \text{B2)} \simeq \mathfrak{aff}(\mathbb{R}) \times \mathfrak{su}(2),$$

where respectively

$$\text{B1)} \quad \begin{cases} \mathfrak{aff}(\mathbb{R}) \simeq \text{Span}\{a_1 e_1 + e_5, f_4 e_1 + e_5\} \\ \mathfrak{su}(2) \simeq \text{Span}\{e_3, e_4, e_5\} \end{cases}$$

and

$$\text{B2)} \begin{cases} \mathfrak{aff}(\mathbb{R}) \simeq \text{Span}\{a_1 e_1 + b_1 e_2 + 2e_5, e_1 + c_3 e_5\} \\ \mathfrak{su}(2) \simeq \text{Span}\{e_3, e_4, e_5\}. \end{cases}$$

Again in the cases B3) and B4) \mathfrak{g} is solvable and

$$\text{B3), B4)} \cong \mathbb{R}^2 \ltimes \mathfrak{h}_3$$

by using for B3) the new basis

$$\left\{ G_1 = a_1 e_1 + 2e_5, G_2 = \frac{1}{a_1} e_2, G_j = e_j, j = 3, 4, 5 \right\},$$

with $\mathbb{R}^2 = \text{Span}\{G_2, G_5\}$, $\mathfrak{h}_3 = \text{Span}\{G_1, G_3, G_4\}$ and

$$\text{ad}_{G_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{f_4}{a_1} \\ 0 & -\frac{f_4}{a_1} & \frac{1}{2} \end{pmatrix}, \quad \text{ad}_{G_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

For B4) we may choose the new basis

$$\left\{ H_1 = a_1 e_1 + b_1 e_2 + 2e_5, H_2 = \frac{1}{b_1} e_1, H_j = e_j, j = 3, 4, 5 \right\},$$

with $\mathbb{R}^2 = \text{Span}\{H_2, H_5\}$, $\mathfrak{h}_3 = \text{Span}\{H_1, H_3, H_4\}$ and

$$\text{ad}_{H_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{c_3}{b_1} \\ 0 & \frac{c_3}{b_1} & -\frac{1}{2} \end{pmatrix}, \quad \text{ad}_{H_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In the cases A3), A4), B3), B4), the corresponding Lie algebras are all isomorphic to a semidirect product $\mathfrak{g}_t = \mathbb{R}^2 \ltimes_{\psi_t} \mathfrak{h}_3$, where $\mathbb{R}^2 = \text{Span}\{X, Y\}$, $\mathfrak{h}_3 = \text{Span}\{v_1, v_2, v_3\}$ and

$$[v_2, v_3] = -v_1, \quad \psi_t(X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & t \\ 0 & -t & \frac{1}{2} \end{pmatrix}, \quad \psi_t(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By changing X to $X + tY$ we obtain an isomorphism between \mathfrak{g}_t and \mathfrak{g}_0 with structure equations (1).

To summarize we can state the following

Theorem 4.4. *If a 5-dimensional Sasakian Lie algebra \mathfrak{g} has trivial center, then it is isomorphic to one of the following Lie algebras: the direct products $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$, or the non-unimodular solvable Lie algebra \mathfrak{g}_0 .*

Remark 4.5. Note that \mathfrak{g}_0 corresponds to the Lie algebra numbered 22 in the classification of 5-dimensional solvable contact Lie algebras provided by Diatta in [14].

5. 5-DIMENSIONAL SASAKIAN α -EINSTEIN LIE ALGEBRAS

In this section we study Sasakian α -Einstein Lie algebras. A Sasakian Lie algebra $(\mathfrak{g}, \Phi, \alpha, \xi, g)$ is called α -Einstein if the Ricci tensor Ric_g of the metric g satisfies $\text{Ric}_g = \lambda g + \nu \alpha \otimes \alpha$ for some $\lambda, \nu \in \mathbb{R}$.

It is known that the canonical Sasakian structure on \mathfrak{h}_5 is α -Einstein. Furthermore, in view of [1, Proposition 4.2] the Lie algebra \mathfrak{g}_0 from Theorem 4.4 is the only solvable (non nilpotent) 5-dimensional Lie algebra admitting a Sasakian α -Einstein structure. Thus, in order to determine all the 5-dimensional Lie algebras admitting such a structure, we only have to consider the non-solvable ones, which are $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ and $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$ according to Theorems 4.1 and 4.4.

Proposition 5.1. *The Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ admits Sasakian α -Einstein structures, while there are none Sasakian α -Einstein structures on $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$.*

Proof. All Sasakian structures on $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ are described in the cases A1), A2) given in §4.2. Performing standard computations we obtain that in the case A1) the Ricci tensor is given by the following matrix

$$\text{Ric}_g = \begin{pmatrix} -\frac{2c_3^2+2}{c_3^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2c_3^2+2}{c_3^2} & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Therefore for $c_3 = \pm 1$ we see that the Sasakian structure is α -Einstein. In the case A2) we have

$$\text{Ric}_g = \begin{pmatrix} -(2 + a_1^2 + b_1^2) & 0 & 0 & 0 & 0 \\ 0 & -(2 + a_1^2 + b_1^2) & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Therefore for $a_1^2 + b_1^2 = 2$ we obtain that the Sasakian structure is α -Einstein.

We consider now the Lie algebra $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$. For this algebra the Sasakian structures are described by the cases B1), B2) studied in §4.2. In the case B1) the Ricci tensor is given by

$$\text{Ric}_g = \begin{pmatrix} -(2 + a_1^2) & 0 & 0 & 0 & 0 \\ 0 & -(2 + a_1^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix},$$

whereas in the case B2) it is given by

$$\text{Ric}_g = \begin{pmatrix} -(2 + a_1^2 + b_1^2) & 0 & 0 & 0 & 0 \\ 0 & -(2 + a_1^2 + b_1^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

As a consequence the Sasakian structures on this Lie algebra never satisfy the α -Einstein condition. \square

To sum up, we can now state the following

Theorem 5.2. *The only 5-dimensional Lie algebras admitting a Sasakian α -Einstein structure are \mathfrak{h}_5 , \mathfrak{g}_0 and $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$.*

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